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1992 J. Phys. A: Math. Gen. 25 5999

(http://iopscience.iop.org/0305-4470/25/22/024)

J. Phys. A: Math. Gen. 25 (1992) 5999–6003. Printed in the UK

Enumeration of the order-14 invariants formed from the Riemann tensor

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Received 1 May 1992

Abstract. The results of a complete enumeration of the scalars formed from the Riemann tensor by covariant differentiation, multiplication and contraction of order 14 in the derivatives of the metric are presented. The corresponding enumeration for the numbers of scalars constructed solely from the Weyl tensor is also given.

1. Introduction

Fulling *et al* [1] (referred to herein as I) have recently discussed the enumeration of the scalars formed from the Riemann tensor (of a torsionless, metric-compatible connection) by covariant differentiation, multiplication and contraction. They have determined the number of independent homogeneous scalar monomials of each order and degree up to order 12 in derivatives of the metric. In this paper we give the corresponding enumeration for order 14. The technical background has been given in I, and references therein, and will not be repeated here.

While it is hardly practical, possibly not interesting, to explicitly construct the scalars of high order there is some interest to know how the number of scalars grows with increasing order with the hope that eventually one may develop asymptotic theories with the given results supplying test points in much in way that MacMahon's enumeration of $p(n)$, the number of ordered partitions of the integer n, to $n=200$ played in the development of Hardy and Ramanujan's asymptotic form [2] that for sufficiently large n gave an exact result. The total number of Riemann scalars for order $2n$ grows faster than $n+1!$ and rapidly outgrows the possibility of explicit computer enumeration. The step from the twelfth order reported in I to fourteenth order is substantial and required a number of special considerations that will be briefly reviewed before presenting the final results.

The key tool used in making the calculations has been the mathematical operation known as plethysm [2–6] and the properties of symmetric functions [7]. In general we follow the definitive notation of Macdonald [7] for symmetric functions, partitions etc. and the notation specified in I. The practical calculations were all performed using the interactive programme SCHUR©.

2. Enumeration of Riemann scalars

As shown in I the master object for enumerating the Riemann scalars is

$$\mathscr{G} = \sum_{m=1}^{\infty} (t^2\{2^2\} + t^3\{32\} + t^4\{42\} + \ldots)^{\otimes m}. \tag{2.1}$$

There is a Riemann scalar for every S-function $\{\lambda\}$ arising in the evaluation of (2.1) whose partition label $\lambda \equiv \lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_p$ involves only *even* parts. The evaluation of the Riemann scalars of order n involves the resolution of all plethysms and outer S-function products associated with t^n where n is necessarily even. This is accomplished in the following algorithmic manner:

(1) List all the ordered partitions of n having no part equal to unity.
(2) For each partition $(\lambda_p^{m_p}, \ldots, \lambda_i^{m_i}, \ldots, 2^{m_2}, 1^{m_1})$, with m_i being the number of times the part λ_i is repeated, replace each $\lambda_i^{m_i}$ by $\{\lambda_i, 2\} \otimes \{m_i\}$ noting that $\{\lambda_i, 2\} \otimes \{1\} \equiv \{\lambda_i, 2\}$.
(3) For each partition replacement evaluate the plethysms as lists of S-functions and then combine the lists under outer S-function multiplication using the Littlewood–Richardson rule.
(4) Remove from the resulting lists all partitions involving any odd parts.
(5) In arbitrarily high dimensions there is one independent Riemann scalar of order n for each surviving S-function.
(6) The minimal dimension capable of supporting a given independent Riemann scalar associated with an S-function indexed by a p part partition (λ) is p. Thus for order 14 step (1) leads to the 34 partitions:

$$(14), (12\,2), (11\,3), (10\,4), (10\,2^2), (95), (932), (86), (842), (83^2), (82^3), (7^2),$$
$$(752), (743), (732^2), (6^2\,2), (653), (64^2), (642^2), (63^22), (62^4), (5^2\,4), (5^2\,2^2),$$
$$(5432), (53^3), (532^3), (4^3\,2), (4^2\,3^2), (4^2\,2^3), (43^2\,2^2), (42^5), (3^4\,2), (3^2\,2^4), (2^7). \tag{2.2}$$

Step (2) involves, for example, the replacement of the partition $(43^2\,2^2)$ by $\{42\}\cdot(\{32\}\otimes\{2\})\cdot(\{2^2\}\otimes\{2\})$ while step (3) involves explicit evaluation of the plethysms and outer S-function multiplications to yield a total of 674 713 S-functions. Step (5) reduces this list to 22 907 S-functions and hence we may conclude that there are 22 907 independent Riemann scalars associated with the partition $(43^2\,2^2)$. The partitions give a convenient way of classifying the multitudinous Riemann scalars. Among the 22 907 scalars there are 9273 involving S-functions indexed by partitions having five parts and hence that set of 9273 scalars can only exist within a minimal dimension of five and higher. The complete set of results is given in table 1. The table presents in the first vertical column the class (or partition) from which the scalars are derived. The numerical column headings give the minimal dimensions at and above which the scalars are independent.

The main difficulty in establishing the table was the evaluation of the terms in the plethysm $\{2^2\}\otimes\{7\}$. This was done by first noting that

$$\{\lambda\}\otimes\{n\} = \sum_{r=1}^{n-1} K_{\lambda\mu} m_{\mu\cdot(n-r)} \cdot \{\lambda\}\otimes\{r\} \tag{2.3}$$

where $K_{\lambda\mu}$ is the Kostka matrix that converts the S-function $\{\lambda\}$ into a sum of monomials m_μ indexed by the partitions μ [7]. The index $\mu\cdot(n-r)$ is to be interpreted as multiplying each part of the partition μ by the integer $(n-r)$. Equation (2.3) is, like most other algorithms for evaluting plethysms, recursive. The monomials in (2.3) were multiplied by the S-functions to yield lists of S-functions via Gordan's formula as outlined recently by Carré [8]. Like all other techniques for evaluating plethysms, there is considerable overcounting making memory issues of significance. Memory issues for the particular case of $\{2^2\}\otimes\{7\}$ were addresssed by removing partitions having odd

Table 1. The count of Riemann sclars for order 14.

Class	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
14	1	—	—	—	—	—	—	—	—	—	—	—	—	1
12 2	1	2	1	—	—	—	—	—	—	—	—	—	—	4
11 3	1	3	1	—	—	—	—	—	—	—	—	—	—	5
10 4	2	6	2	—	—	—	—	—	—	—	—	—	—	10
$10\,2^2$	1	7	10	4	1	—	—	—	—	—	—	—	—	23
95	2	7	2	—	—	—	—	—	—	—	—	—	—	11
932	1	19	29	10	1	—	—	—	—	—	—	—	—	60
86	3	10	3	—	—	—	—	—	—	—	—	—	—	16
842	2	34	54	19	2	—	—	—	—	—	—	—	—	111
83^2	2	27	43	15	2	—	—	—	—	—	—	—	—	89
82^3	1	15	50	47	18	4	1	—	—	—	—	—	—	136
7^2	3	8	3	—	—	—	—	—	—	—	—	—	—	14
752	2	44	70	23	2	—	—	—	—	—	—	—	—	141
743	3	73	126	41	3	—	—	—	—	—	—	—	—	246
732^2	1	60	278	270	87	13	1	—	—	—	—	—	—	710
$6^2\,2$	3	32	53	18	3	—	—	—	—	—	—	—	—	109
653	4	94	169	55	4	—	—	—	—	—	—	—	—	326
64^2	4	65	124	44	5	—	—	—	—	—	—	—	—	242
642^2	2	99	482	489	162	25	2	—	—	—	—	—	—	1 261
$63^2\,2$	2	129	694	724	235	31	2	—	—	—	—	—	—	1 817
62^4	1	25	187	368	266	86	20	4	1	—	—	—	—	958
$5^2\,4$	4	72	141	49	5	—	—	—	—	—	—	—	—	271
$5^2\,2^2$	2	67	314	317	110	18	2	—	—	—	—	—	—	830
5432	3	328	1 987	2 186	705	81	3	—	—	—	—	—	—	5 293
53^3	2	88	516	591	198	26	2	—	—	—	—	—	—	1 423
532^3	1	109	1 282	3 007	2 180	651	106	13	1	—	—	—	—	7 350
$4^3\,2$	2	81	465	533	190	28	3	—	—	—	—	—	—	1 302
$4^2\,3^2$	3	170	1 021	1 207	428	58	4	—	—	—	—	—	—	2 891
$4^2\,2^3$	2	86	923	2 118	1 616	510	100	14	2	—	—	—	—	5 371
$43^2\,2^2$	2	259	3 646	9 273	7 142	2 206	345	32	2	—	—	—	—	22 907
42^5	1	26	384	1 526	2 058	1 171	360	81	18	4	1	—	—	5 630
$3^4\,2$	1	71	955	2 490	1 996	636	104	10	1	—	—	—	—	6 264
$3^2\,2^4$	1	63	1 261	5 639	7 915	4 537	1 341	260	44	7	1	—	—	21 069
2^7	1	7	80	343	709	621	328	105	36	10	4	1	1	2 246
Totals	67	2 186	15 356	31 406	26 043	10 702	2 724	519	105	21	6	1	1	89 137

parts at the end of each monomial multiplication and thus only the even part partition indexes of the S-functions were retained, a method appropriate for the case in hand but still requiring the complete construction of all the lower-order plethysms. All the operations were carried out using appropriate commands in SCHUR.

3. Enumeration of Weyl scalars

The master object $\mathscr{W}$ for the enumeration of scalars arising from tensor polynomials in the Weyl tensor is of the same form as for (2.1) except for the replacement of $\{k+2, 2\}$ by $[k+2, 2]$ where the square brackets $[\lambda]$ label irreducible representations of the full orthogonal group O_d [9] and hence

$$\mathscr{W} \equiv \sum_{m=1}^{\infty} (t^2[2^2] + t^3[32] + t^4[42] + \ldots)^{\otimes m}. \tag{3.1}$$

There will be a Weyl scalar for every occurrence of the O_d identity representation [0] and hence in principle one need only count the number of scalars arising from (3.1). However, as with the Riemann scalars, there exists a minimal dimension for each scalar. In practice the set of scalars is constructed for a dimension d large enough to ensure that every irreducible represention $[\lambda]$ of O_d arising in (3.1) at order n is standard, which is certainly the case for $d \geqslant 2n$. The number of scalars associated with the minimal dimension m, at and above which the scalars are linearly independent, is determined by taking the set of irreducible representations from (3.1) and determining the number of scalars $n(m)$ and $n(m+1)$ for O_m and O_{m+1} after application of the orthogonal group modification rules. The number $n(m+1)-n(m)$ is then the number of linearly independent Weyl scalars that exist in dimension m and above, but not in dimension $<m$. The Weyl scalars were enumerated by first noting that in O_d a scalar [0] (but not a pseudo-scalar $[0]^*$) is only possible in the Kronecker product $[\lambda]\times[\mu]$ if $[\lambda]\equiv[\mu]$. Thus to count the number of scalars in $[42]\cdot[2^2]\otimes\{5\}$ it suffices to count the number of times [42] occurs in $[2^2]\otimes\{5\}$. These operations were all carried out with standard SCHUR commands. The only difficult terms were those associated with $[2^2]\otimes\{7\}$. These were enumerated by first noting that if $[\alpha]$ and $[\beta]$ are two lists of

Table 2. The count of the Weyl scalars of order 14.

Class	4	5	6	7	8	9	10	11	12	13	14	Total
7^2	1	—	—	—	—	—	—	—	—	—	—	1
752	0	1	—	—	—	—	—	—	—	—	—	1
743	0	1	—	—	—	—	—	—	—	—	—	1
732^2	2	1	1	—	—	—	—	—	—	—	—	4
$6^2 2$	2	1	1	—	—	—	—	—	—	—	—	4
653	2	3	1	—	—	—	—	—	—	—	—	6
64^2	2	2	1	—	—	—	—	—	—	—	—	5
642^2	1	16	7	2	—	—	—	—	—	—	—	26
$63^2 2$	3	21	10	2	—	—	—	—	—	—	—	36
62^4	2	7	12	3	1	—	—	—	—	—	—	25
$5^2 4$	2	4	1	—	—	—	—	—	—	—	—	7
$5^2 2^2$	7	18	15	3	1	—	—	—	—	—	—	44
5432	15	91	70	13	1	—	—	—	—	—	—	190
53^3	3	25	21	4	1	—	—	—	—	—	—	54
532^3	4	92	161	13	2	—	—	—	—	—	—	378
$4^3 2$	2	26	21	5	1	—	—	—	—	—	—	55
$4^2 3^2$	13	67	60	16	2	—	—	—	—	—	—	158
$4^2 2^3$	13	90	179	85	24	3	1	—	—	—	—	395
$43^2 2^2$	23	385	774	409	90	10	1	—	—	—	—	1692
42^5	0	63	204	205	77	19	3	1	—	—	—	572
$3^4 2$	8	113	246	141	37	5	1	—	—	—	—	551
$3^2 2^4$	13	940	953	952	405	94	18	3	1	—	—	2705
2^7	3	20	107	136	111	37	16	4	2	0	1	437
Totals	121	1313	2845	1989	753	168	40	8	3	0	1	7347

O_d irreducible representations then

$$[(\alpha \cdot \beta)/D] \equiv [\alpha/D] \times [\beta/D] \tag{3.2}$$

where D is the usual S-function series [9]. Thus to obtain the scalars in $[2^2] \otimes \{n\}$ write

$$[2^2] \otimes \{n\} = \sum_{m=0}^{n} (-1)^m [(\{2^2\} \otimes \{n-m\} \cdot \{2\} \otimes \{1^m\})/D] \tag{3.3a}$$

$$= \sum_{m=0}^{n} (-1)^m [(\{2^2\} \otimes \{n-m\})/D] \cdot [(\{2\} \otimes \{1^m\})/D]. \tag{3.3b}$$

For each term in (3.3b) the two plethysms are constructed as two lists, each list is skewed with the D-series and the two resulting lists modified for the group O_d of interest and the two lists compared for common terms to obtain the number of scalars for the term. The process is repeated for each term in succession to finally yield the total count of the Weyl scalars. This approach is more efficient than the direct approach since it avoids the need to form products of large lists of O_n irreducible representations. Had one attempted to evaluate all the terms in $[2^2] \otimes \{7\}$ directly one would have been faced with enumerating over 250 000 000 irreducible representations of O_n to finally discover the 437 Weyl scalars associated with that plethysm.

The complete list of Weyl scalars at order 14 are collected together in table 2 using the same notation as for table 1.

4. Conclusion

The scalars associated with the Riemann and Weyl tensors have been enumerated, and their minimal dimensions established, for the fourteenth order. The Riemann scalars are much more numerous that the Weyl scalars, which is no surprise; however, paradoxically the latter requires the enumeration of plethysms involving many more partitions. Thus whereas $\{32\} \otimes \{4\}$ involves 3131 terms the O_d plethysm $[32] \otimes \{4\}$ involves 373 689 terms. As of yet there seems no way of permitting a direct enumeration that avoids gross overcounting. Unless a radically new procedure of enumerating invariants is discovered, even with improvements in computer memories and speed further progress is likely to be limited to only a very few higher orders. A more fruitful approach is likely to be the study of the stability properties of plethysms, a far from trivial subject, with the aim of achieving asymptotic formulae. Enough has been achieved to create testing points for such attempts.

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